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Geometric Phase in a Generalized Jaynes-Cummings Model with Double Mode Operators and Phase Operators

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Abstract By using the Lewis-Riesenfeld invariant theory, the geometric phase in a generalized Jaynes-Cummings model with double mode operators and phase operators has been studied. Compared with the dynamical phase, the geometric phase in a cycle case is independent of the frequency of the double photon field, the coupling coefficient between photons and atoms, and the atom transition frequency. It is apparent that the geometric phase has the pure geometric and topological characteristics, which means that the geometric phase represents the holonomy in the Hermitian linear bundles.

Keywords Geometric phase · Generalized Jaynes-Cummings model

1 Introduction

It is known that the concept of geometric phase was first introduced by Pancharatnam [1] in studying the interference of classical light in distinct states of polarization. Berry [2] found the quantal counterpart of Pancharatnam's phase in the case of cyclic adiabatic evolution. The extension to non-adiabatic cyclic evolution was developed by Aharonov and Anandan [3]. Samuel et al. [4] generalized the pure state geometric phase by extending it to non-cyclic evolution and sequential projection measurements. The geometric phase is a consequence of quantum kinematics and is thus independent of the detailed nature of the dynamical origin of the path in state space. Mukunda and Simon [5] gave a kinematic approach by taking the path traversed in state space as the primary concept for the geometric phase. Further generalizations and refinements, by relaxing the conditions of adiabaticity, unitarity, and cyclicity of the evolution, have since been carried out [6]. Recently, the geometric phase of the mixed states has also been studied [7–9].

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As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [10] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized by introducing the concept of basic invariants and used to study the geometric phases [11–14] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations, but also provides us with new insights in many physical phenomena. The concept of Berry's phase has been developed in some different directions [15–27]. In this letter, by using the Lewis-Riesenfeld invariant theory, we shall study the geometric phase in a generalized Jaynes-Cummings model with double mode operators and phase operators.

2 Model

The Hamiltonian in a generalized Jaynes-Cummings model with double mode operators and phase operators is described as [28, 29]

$$\hat{H} = \omega(t)(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + \frac{1}{2}\Omega\hat{\sigma}_z + \lambda(t)\left[\hat{\sigma}_+\sqrt{\frac{\hat{a}-\hat{b}^\dagger}{\hat{a}^\dagger-\hat{b}}}\sqrt{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}} + \sqrt{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}}\sqrt{\frac{\hat{a}^\dagger-\hat{b}}{\hat{a}-\hat{b}^\dagger}}\hat{\sigma}_-\right], \tag{1}$$

where $\hat{a}(\hat{a}^\dagger)$ and $\hat{b}(\hat{b}^\dagger)$are the double photon annihilation (creation) operators. $\hat{\sigma}_+(\hat{\sigma}_-)$ are the pseudospin operators for the atom defined as $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$ with $\hat{\sigma}_x$ and $\hat{\sigma}_y$ being the Pauli matrices, $\lambda(t)$ stands for the coupling strength between the atom and the light field, $\omega(t)$ is the frequency of the electromagnetic wave, and Ω is the transition frequency of the atom.

We introduce the following operators

$$\hat{Q}_+ = \hat{\sigma}_+\sqrt{\frac{\hat{a}-\hat{b}^\dagger}{\hat{a}^\dagger-\hat{b}}}\sqrt{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}}, \qquad \hat{Q}_- = \sqrt{\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}}\sqrt{\frac{\hat{a}^\dagger-\hat{b}}{\hat{a}-\hat{b}^\dagger}}\hat{\sigma}_-, \tag{2}$$

$$\hat{\hat{M}} = \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b} + \frac{1}{2}\hat{\sigma}_z + \frac{1}{2}, \tag{3}$$

one has the following commutation relations

$$[\hat{Q}_+, \hat{Q}_-] = \hat{\hat{M}}\hat{\sigma}_z, \quad [\hat{\hat{M}}, \hat{Q}_-] = [\hat{\hat{M}}, \hat{Q}_+] = 0, \quad \hat{Q}_-^2 = \hat{Q}_+^2 = 0, \tag{4}$$

$$\{\hat{Q}_-, \hat{\sigma}_z\} = \{\hat{Q}_+, \hat{\sigma}_z\} = 0, \quad \hat{\hat{M}} = \{\hat{Q}_+, \hat{Q}_-\}, \quad (\hat{Q}_+ - \hat{Q}_-)^2 = -\hat{\hat{M}}, \tag{5}$$

$$(\hat{Q}_+ - \hat{Q}_-)\hat{\sigma}_z = -(\hat{Q}_+ + \hat{Q}_-), \quad [\hat{Q}_+, \hat{\sigma}_+] = [\hat{Q}_-, \hat{\sigma}_-] = 0, \quad [\hat{Q}_-, \hat{\sigma}_z] = 2\hat{Q}_-, \tag{6}$$

$$[\hat{Q}_+, \hat{\sigma}_z] = -2\hat{Q}_+, \quad [\hat{Q}_-, \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}] = -\hat{Q}_-, \quad [\hat{Q}_+, \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}] = \hat{Q}_+, \tag{7}$$

where {, } stands for the anticommuting bracket. Equation (1) becomes

$$\hat{H} = \omega(t)(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}) + \frac{1}{2}\Omega\hat{\sigma}_z + \lambda(t)[\hat{Q}_+ + \hat{Q}_-]. \tag{8}$$

It is easy to find that

$$\hat{\bar{M}}\frac{1}{\sqrt{2}}\begin{pmatrix}|n-1,m\rangle\\|n,m\rangle\end{pmatrix} = (n-m)\frac{1}{\sqrt{2}}\begin{pmatrix}|n-1,m\rangle\\|n,m\rangle\end{pmatrix}, \tag{9}$$

so we can restrict our study in the sub-Hilbert space of the supersymmetric quasi-algebra $\hat{\bar{M}}$, $\hat{Q}_\pm$, $\hat{\sigma}_\pm$, $\hat{\sigma}_z$ and $\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}$. Below, we replace operator $\hat{\bar{M}}$ with the particular eigenvalue $\bar{M} = (n-m)$.

3 Dynamical and Geometric Phases

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [10]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{\varrho}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial\hat{\varrho}(t)}{\partial t} + [\hat{\varrho}(t), \hat{H}(t)] = 0. \tag{10}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{\varrho}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{11}$$

where $\frac{\partial\lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{12}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (12) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{\varrho}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{13}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (12). Then the general solution of the Schrödinger equation (12) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{14}$$

where

$$\delta_n(t) = \int_0^t dt'\langle\lambda_n, t'|i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t'\rangle, \tag{15}$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

For the system described by Hamiltonian (8), we can define the following invariant

$$\hat{\varrho}(t) = \alpha(t)\hat{Q}_- + \alpha^*(t)\hat{Q}_+ + \beta(t)\hat{\sigma}_z. \tag{16}$$

Substituting (8) and (16) into (10), one has the auxiliary equations

$$i\dot{\alpha}(t)+\alpha(t)[\Omega-\omega(t)]-2\lambda(t)\beta(t)=0, \tag{17}$$

$$i\dot{\beta}(t)+\lambda(t)\bar{M}[\alpha^*(t)-\alpha(t)]=0, \tag{18}$$

where dot denotes the time derivative, and $\bar{M}$ denotes the eigenvalue of operator $\hat{\bar{M}}$.

In order to obtain a time-independent invariant, we can introduce the unitary transformation operator $\hat{V}(t)=\exp[\xi(t)\hat{Q}_- - \xi^*(t)\hat{Q}_+]$. It is easy to find that when satisfying the following relations

$$\sin(2\sqrt{\bar{M}}|\xi(t)|)=\frac{\sqrt{\bar{M}}[\alpha(t)\xi^*(t)+\alpha^*(t)\xi(t)]}{2|\xi(t)|},\quad \beta(t)=\cos(2\sqrt{\bar{M}}|\xi(t)|), \tag{19}$$

and

$$\begin{aligned}&\frac{\alpha(t)}{2}[1+\cos(2\sqrt{\bar{M}}|\xi(t)|)]-\frac{\beta(t)\xi(t)}{\sqrt{\bar{M}}|\xi(t)|}\sin(2\sqrt{\bar{M}}|\xi(t)|)\\&\quad-\frac{\alpha^*(t)\xi^2(t)}{2|\xi(t)|^2}[1-\cos(2\sqrt{\bar{M}}|\xi(t)|)]=0,\end{aligned} \tag{20}$$

then a time-independent invariant appears

$$\hat{\varrho}_V \equiv \hat{V}^\dagger(t)\hat{\varrho}(t)\hat{V}(t)=\hat{\sigma}_z. \tag{21}$$

According to (19), we can select

$$\xi(t)=\frac{\theta(t)}{\sqrt{2\bar{M}}}\exp[i\gamma(t)],\quad \alpha(t)=\frac{\sin\theta(t)}{\sqrt{2\bar{M}}}\exp[i\gamma(t)],\quad \theta(t)=2\sqrt{\bar{M}}|\xi(t)|. \tag{22}$$

From (22), the invariant $\hat{\varrho}(t)$ in (16) becomes

$$\hat{\varrho}(t)=\frac{\sin\theta(t)}{\sqrt{2\bar{M}}}\{\exp[i\gamma(t)]\hat{Q}_- + \exp[-i\gamma(t)]\hat{Q}_+\}+\cos\theta(t)\hat{\sigma}_z. \tag{23}$$

By using the Baker-Campbell-Hausdoff formula [30]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}=\frac{\partial\hat{L}}{\partial t}+\frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right]+\frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right]+\frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t},\hat{L}\right],\hat{L}\right],\hat{L}\right]+\cdots, \tag{24}$$

with $\hat{V}(t)=\exp[\hat{L}(t)]$, it is easy to find that when satisfying the following equations

$$\begin{aligned}&\lambda(t)[\cos\theta(t)\cos\gamma(t)+i\sin\gamma(t)]+\frac{1}{2\sqrt{\bar{M}}}[\omega(t)-\Omega]\sin\theta(t)\\&\quad+\frac{1}{\sqrt{2\bar{M}}}[i\dot{\theta}(t)+\dot{\gamma}(t)\theta(t)]-\frac{\dot{\gamma}}{\sqrt{\bar{M}}}[\theta(t)-\sin\theta(t)]=0,\end{aligned} \tag{25}$$

one has

$$\hat{H}_V(t)=\hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t)-i\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t}$$

$$= \left\{\lambda(t)\sqrt{2\bar{M}}\cos\gamma(t)\sin\theta(t) + \frac{1}{2}[\Omega - \omega(t)]\cos\theta(t) + \frac{\omega(t)}{2}\right\}\hat{\sigma}_z$$
$$+ \dot{\gamma}(t)[1-\cos\theta(t)]\hat{\sigma}_z + \omega(t)(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}). \tag{26}$$

For $\sigma_z = +1$, one has the particular solution of (12):

$$|\psi_{\sigma_z=+1}(t)\rangle = \exp\left\{-i\int_0^t [\dot{\delta}^d_{\sigma_z=+1}(t') + \dot{\delta}^g_{\sigma_z=+1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix}|n-1,m\rangle \\ 0\end{pmatrix}, \tag{27}$$

where

$$\delta^d_{\sigma_z=+1}(t') = \lambda(t)\sqrt{2(n-m)}\cos\gamma(t)\sin\theta(t)$$
$$+ \frac{1}{2}[\Omega - \omega(t)]\cos\theta(t) + \frac{\omega(t)}{2} + \omega(t)(n-m), \tag{28}$$

$$\dot{\delta}^g_{\sigma_z=+1}(t') = \dot{\gamma}(t')[1-\cos\theta(t')]. \tag{29}$$

For $\sigma_z = -1$, one has

$$|\psi_{\sigma_z=-1}(t)\rangle = \exp\left\{-i\int_0^t [\dot{\delta}^d_{\sigma_z=-1}(t') + \dot{\delta}^g_{\sigma_z=-1}(t')]dt'\right\}\hat{V}(t)\begin{pmatrix}0 \\ |n,m\rangle\end{pmatrix}, \tag{30}$$

where

$$\dot{\delta}^d_{\sigma_z=-1}(t') = -\lambda(t)\sqrt{2(n-m)}\cos\gamma(t)\sin\theta(t)$$
$$- \frac{1}{2}[\Omega - \omega(t)]\cos\theta(t) - \frac{\omega(t)}{2} + \omega(t)(n-m), \tag{31}$$

$$\dot{\delta}^g_{\sigma_z=-1}(t') = -\dot{\gamma}(t')[1-\cos\theta(t')]. \tag{32}$$

From (28)–(29), (31)–(32) we conclude that the dynamical and the geometric phase factors of the system are $\exp[-i\int_0^t \dot{\delta}^d_{\sigma_z}(t')dt']$ and $\exp[-i\int_0^t \dot{\delta}^g_{\sigma_z}(t')dt']$ with $\sigma_z = \pm 1$, respectively. In particular, when we consider a cycle in the parameter space of the invariant $\hat{\varrho}(t)$ and let $\theta(t)$=constant, one has from (29) and (32), respectively

$$\delta^g_{\sigma_z}(T) = \begin{cases} 2\pi(1-\cos\theta) & (\sigma_z = +1), \\ -2\pi(1-\cos\theta) & (\sigma_z = -1). \end{cases} \tag{33}$$

Here $2\pi(1-\cos\theta)$ denotes the solid angle over the parameter space of the invariant $\hat{\varrho}(t)$.

4 Conclusions

In this letter, by using the Lewis-Riesenfeld invariant theory, the geometric phase in a generalized Jaynes-Cummings model with double mode operators and phase operators has been studied. Compared with the dynamical phase, the geometric phase in a cycle case is independent of the frequency of the double photon field, the coupling coefficient between photons and atoms, and the atom transition frequency. It is apparent that the geometric phase has the pure geometric and topological characteristics, which means that the geometric phase represents the holonomy in the Hermitian linear bundles.

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